EXERCISES IN TENSOR ALGEBRA: GEOMETRIC INTERPRETATIONS

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Notations

In the following notes:

- $\mathbb{R}^3$ is the three-dimensional Euclidean space;
- $\mathcal{V}$ is the inner-product linear space of translations of $\mathbb{R}^3$;
- $\mathcal{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ denotes a (positively oriented) basis of $\mathcal{V}$, hence $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are linearly independent unit vectors, pairwise orthogonal;
- "i.e." is the abbreviation for the Latin "id est", which means "that is";
- "e.g." is the abbreviation for the Latin "exempli gratia", which means "for example";
- "w.r.t." is the abbreviation for the English "with respect to".

The "***Why?***" sections contain some detailed proof of what is said in the other parts of these notes. In some cases, the "***Why?***" sections are long and boring. Don't be afraid: you don't have to study and recall them, but you can read them if you are interested in some deeper explanation of the results presented here.

1. Inner product

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two translations in $\mathcal{V}$. By using the basis $\mathcal{B}$, we write both $\boldsymbol{u}$ and $\boldsymbol{v}$ in *Cartesian components*:

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3,$$
$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3,$$

where $u_i, v_i \in \mathbb{R}$ for $i = 1, 2, 3$.

Definition 1. The *inner product* (or *dot product*) between $\boldsymbol{u}$ and $\boldsymbol{v}$ is the real number

$$\boldsymbol{u} \cdot \boldsymbol{v} := u_1v_1 + u_2v_2 + u_3v_3.$$

The inner product between the elements of the basis $\mathcal{B}$ is given by the *Kronecker Delta*:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

whence the Cartesian components of $\boldsymbol{u}$ and $\boldsymbol{v}$ are

$$u_i = \boldsymbol{u} \cdot \boldsymbol{e}_i \quad \text{and} \quad v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i, \qquad \text{for } i = 1, 2, 3.$$

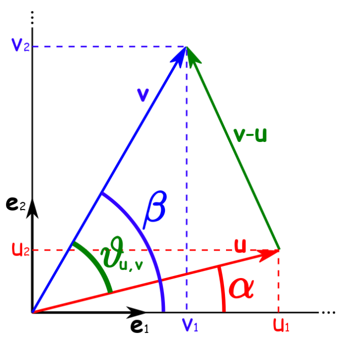
Definition 2. The *length* of a vector $\boldsymbol{u} = u_1 v_1 + u_2 v_2 + u_3 v_3$ is

$$|\boldsymbol{u}| := \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Let $\vartheta_{\boldsymbol{u},\boldsymbol{v}}$ be the (planar) angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Then

$$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\vartheta_{\boldsymbol{u},\boldsymbol{v}}.$$

Why? We can convince ourselves by considering first the dwo-dimensional Euclidean space $\mathbb{R}^2$. In this case, $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2$ and $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2$. Let α and β be the planar angles as in the following picture, and note that $\vartheta_{\boldsymbol{u},\boldsymbol{v}} = \beta - \alpha$.



Then

$$\begin{aligned} u_1 &= |\boldsymbol{u}|\cos\alpha,\\ u_2 &= |\boldsymbol{u}|\sin\alpha,\\ v_1 &= |\boldsymbol{v}|\cos\beta,\\ v_2 &= |\boldsymbol{v}|\sin\beta. \end{aligned}$$

Whence

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{v} &= u_1v_1 + u_2v_2 =\\ &= |\boldsymbol{u}||\boldsymbol{v}|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) =\\ &= |\boldsymbol{u}||\boldsymbol{v}|\cos(\beta-\alpha) =\\ &= |\boldsymbol{u}||\boldsymbol{v}|\cos\vartheta_{\boldsymbol{u},\boldsymbol{v}}. \end{aligned}$$

In $\mathbb{R}^3$ (and, in general, in any $\mathbb{R}^n$), we can consider the vector $\boldsymbol{v}-\boldsymbol{u}$. By the law of cosines,

$$|\boldsymbol{v}-\boldsymbol{u}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}||\boldsymbol{v}|\cos\vartheta_{\boldsymbol{u},\boldsymbol{v}}.$$

On the other hand, by definition of length and by linearity of the inner product,

$$|\boldsymbol{v}-\boldsymbol{u}|^2 = (\boldsymbol{v}-\boldsymbol{u})\cdot(\boldsymbol{v}-\boldsymbol{u}) = \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{u} = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2\boldsymbol{v}\cdot\boldsymbol{u},$$

and hence $\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\vartheta_{\boldsymbol{u},\boldsymbol{v}}$.

2. Representation of a tensor with a matrix (in a fixed basis)

Whenever we fix a basis for the space $\mathcal{V}$ (e.g. the basis $\mathcal{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$) we immediately have that the diads $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ form a basis for the space $L(\mathcal{V})$ of all the (second-order) tensor $\mathbf{L}\colon \mathcal{V} \to \mathbb{R}$, and each tensor $\mathbf{L}$ in $L(\mathcal{V})$ can be written as

$$\mathbf{L} = \sum_{i,j=1}^{3} L_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j,$$

where $L_{ij} := \boldsymbol{e}_i \cdot \mathbf{L}\boldsymbol{e}_j$ are the *Cartesian components* of $\mathbf{L}$.

For this reason we can always represent the tensor $\mathbf{L}$ *with respect to the basis* $\mathcal{B}$ by using the *matrix*

$$[L] := (L_{ij})_{i,j=1,2,3} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1\cdot\mathbf{L}\boldsymbol{e}_1 & \boldsymbol{e}_1\cdot\mathbf{L}\boldsymbol{e}_2 & \boldsymbol{e}_1\cdot\mathbf{L}\boldsymbol{e}_3 \\ \boldsymbol{e}_2\cdot\mathbf{L}\boldsymbol{e}_1 & \boldsymbol{e}_2\cdot\mathbf{L}\boldsymbol{e}_2 & \boldsymbol{e}_2\cdot\mathbf{L}\boldsymbol{e}_3 \\ \boldsymbol{e}_3\cdot\mathbf{L}\boldsymbol{e}_1 & \boldsymbol{e}_3\cdot\mathbf{L}\boldsymbol{e}_2 & \boldsymbol{e}_3\cdot\mathbf{L}\boldsymbol{e}_3 \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{L}\boldsymbol{e}_1 & \mathbf{L}\boldsymbol{e}_2 & \mathbf{L}\boldsymbol{e}_3 \end{matrix}$$

whose columns are the vectors obtained by applying $\mathbf{L}$ to the elements $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of the basis $\mathcal{B}$.

The matrices that represent $\mathbf{L}^\top$ and $\mathbf{L}^{-1}$ w.r.t. the basis $\mathcal{B}$ are, respectively, $[L]^\top$ and $[L]^{-1}$.

Why? If we apply the tensor $\mathbf{L}$ to a generic vector $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$ in $\mathcal{V}$ we obtain

$$\begin{aligned} \mathbf{L}\boldsymbol{u} &= \mathbf{L}\left(\sum_{j=1}^{3} u_j\boldsymbol{e}_j\right) = \sum_{j=1}^{3} u_j\mathbf{L}\boldsymbol{e}_j = \sum_{j=1}^{3}(\boldsymbol{u}\cdot\boldsymbol{e}_j)\mathbf{L}\boldsymbol{e}_j = \\ &= \sum_{j=1}^{3}(\boldsymbol{u}\cdot\boldsymbol{e}_j)\left(\sum_{i=1}^{3}(\mathbf{L}\boldsymbol{e}_j\cdot\boldsymbol{e}_i)\boldsymbol{e}_i\right) = \sum_{i,j=1}^{3}(\boldsymbol{u}\cdot\boldsymbol{e}_j)(\boldsymbol{e}_i\cdot\mathbf{L}\boldsymbol{e}_j)\boldsymbol{e}_i = \\ &= \sum_{i,j=1}^{3}(\boldsymbol{e}_i\cdot\mathbf{L}\boldsymbol{e}_j)(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{u} = \left(\sum_{i,j=1}^{3} L_{i,j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right)\boldsymbol{u}. \end{aligned}$$

Example 1. The matrices that represent the identity tensor $\mathbf{I}$ and the null tensor $\mathbf{0}$ w.r.t *any* basis $\mathcal{B}$ are

$$[I] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad [0] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 2. W.r.t. $\mathcal{B}$, the projections $\mathbf{P}_{||}(\boldsymbol{e}_1) := \boldsymbol{e}_1\otimes\boldsymbol{e}_1$ and $\mathbf{P}(\boldsymbol{e}_1) := \mathbf{I} - \boldsymbol{e}_1\otimes\boldsymbol{e}_1 = \boldsymbol{e}_2\otimes\boldsymbol{e}_2 + \boldsymbol{e}_3\otimes\boldsymbol{e}_3$ are respectively represented by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 3. Let $\mathbf{L}$ be the tensor that maps

$$\boldsymbol{e}_1 \mapsto 2\boldsymbol{e}_1 + 2\boldsymbol{e}_2, \quad \boldsymbol{e}_2 \mapsto -\frac{1}{2}\boldsymbol{e}_1 + \boldsymbol{e}_2 \quad \text{and} \quad \boldsymbol{e}_3 \mapsto \boldsymbol{e}_3.$$

Then, w.r.t. the basis $\mathcal{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the matrices that represent $\mathbf{L}$ and $\mathbf{L}^\top$ are, respectively,

$$[L] = \begin{pmatrix} 2 & -\frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad [L]^\top = \begin{pmatrix} 2 & 2 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order to find the matrix that represents the inverse $\mathbf{L}^{-1}$, we can proceed by computing directly $[L]^{-1}$. We consider a generic matrix

$$[L]^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and we impose the condition $[L]^{-1}[L] = [I]$:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 2 & -\frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We obtain the systems

$$\begin{cases} 2a + 2b = 1 \\ -\dfrac{1}{2}a + b = 0 \\ c = 0 \end{cases} \qquad \begin{cases} 2d + 2e = 0 \\ -\dfrac{1}{2}d + e = 1 \\ f = 0 \end{cases} \qquad \begin{cases} 2g + 2h = 0 \\ -\dfrac{1}{2}g + h = 0 \\ i = 1 \end{cases}$$

with solutions

$$\begin{cases} a = \dfrac{1}{3} \\ b = \dfrac{1}{6} \\ c = 0 \end{cases} \qquad \begin{cases} d = -\dfrac{2}{3} \\ e = \dfrac{2}{3} \\ f = 0 \end{cases} \qquad \begin{cases} g = 0 \\ h = 0 \\ i = 1 \end{cases}$$

Then

$$[L]^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ -\frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Exterior product

A tensor $\mathbf{W}$ is *skew* if $\mathbf{W}^\top = -\mathbf{W}$. When $\mathbf{W}$ is a skew tensor, we can always associate with $\mathbf{W}$ a vector $\boldsymbol{w}(\mathbf{W})$, called the *axial vector* of $\mathbf{W}$, such that

(1) $\mathbf{W}\boldsymbol{w}(\mathbf{W}) = \mathbf{0}$ ($\boldsymbol{w}(\mathbf{W})$ is in the *axis* $\mathcal{A}(\mathbf{W}) := \{\boldsymbol{u} \in \mathcal{V} \colon \mathbf{W}\boldsymbol{u} = \mathbf{0}\}$ of $\mathbf{W}$),

(2) $|\boldsymbol{w}(\mathbf{W})|^2 = \dfrac{|\mathbf{W}|^2}{2} = \dfrac{\operatorname{tr}(\mathbf{W}\mathbf{W}^\top)}{2}$.

Conversely, the *skew tensor associated with* the vector $\boldsymbol{w}$ is the skew tensor $\mathbf{W}(\boldsymbol{w})$ such that $\boldsymbol{w}$ is its axial vector.

Actually, when we consider the skew tensor $\mathbf{W}$ we can always find two different vectors that satisfy the two conditions above (if we call one of them $\boldsymbol{w}$, then the other is $-\boldsymbol{w}$). However, if we choose one of them to be the axial vector of $\mathbf{W}$ and impose the linearity condition $\boldsymbol{w}(\mathbf{W}_1 + \mathbf{W}_2) = \boldsymbol{w}(\mathbf{W}_1) + \boldsymbol{w}(\mathbf{W}_2)$, then the axial vectors of all the skew tensors of $\mathcal{V}$ are automatically determined. This choice is strictly related to the *orientation* of the space $\mathcal{V}$. In these notes (and in the homework) we fix the *positive orientation*, which is the choice of the axial vectors such that

$$\mathbf{W}(\boldsymbol{e}_1)\boldsymbol{e}_2 = \boldsymbol{e}_3.$$

Definition 3. The *exterior product* (or *cross product*) between the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is the vector

$$\boldsymbol{u} \times \boldsymbol{v} := \mathbf{W}(\boldsymbol{u})\boldsymbol{v}.$$

By choosing the positive orientation, we immediately obtain

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3.$$

We also have

$$\boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 \quad \text{and} \quad \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2.$$

Moreover, the exterior product is linear in both the arguments, and $\boldsymbol{v} \times \boldsymbol{u} = -\boldsymbol{u} \times \boldsymbol{v}$.

In Cartesian components we have

$$\boldsymbol{u} \times \boldsymbol{v} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_k \boldsymbol{e}_i,$$

where ε_{ijk} are the component of *Ricci alternator* (or also the *3-dimensional Levi-Civita symbol*)

$$\varepsilon_{ijk} := \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{if } ijk \text{ is not a permutation of } 123 \end{cases}$$

Note. *An* even *permutation of* 123 *is a permutation that can be obtained from* 123 *by an even number of two-element exchanges. An* odd *permutation of* 123 *is a permutation that can be obtained from* 123 *by an odd number of two-element exchanges. Hence, the permutations of* 123 *are*

$$\text{even:} \quad 123 \quad 231 \quad 312, \qquad\qquad \text{odd:} \quad 132 \quad 213 \quad 321.$$

The exterior product can also be computed by using the *formal determinant*

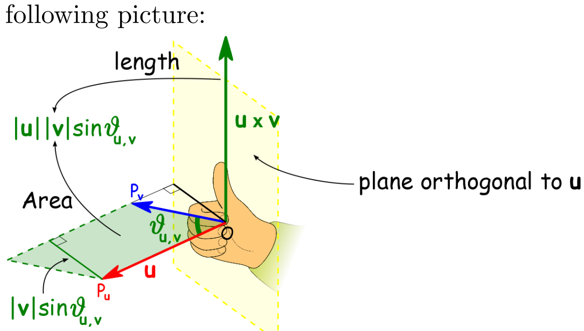
$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3)\boldsymbol{e}_2 + (u_1 v_2 - u_2 v_1)\boldsymbol{e}_3.$$

The vector $\boldsymbol{u} \times \boldsymbol{v}$ enjoys the following two interesting properties:

(1) it is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$,
(2) its length is the area of the parallelogram"described by" $\boldsymbol{u}$ and $\boldsymbol{v}$, whose vertices are the points O, $P_{\boldsymbol{u}} := O + \boldsymbol{u}$, $P_{\boldsymbol{v}} := O + \boldsymbol{v}$ and $P_{\boldsymbol{u}+\boldsymbol{v}} := O + \boldsymbol{u} + \boldsymbol{v}$:

$$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \sin \vartheta_{\boldsymbol{u},\boldsymbol{v}}.$$

Moreover, in order to find the orientation of $\boldsymbol{u} \times \boldsymbol{v}$ we can use the right-hand rule, as shown in the following picture:



Why? The first property ($\boldsymbol{u}\times\boldsymbol{v}$ orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$) follows immediately from the fact that $\mathbf{W}(\boldsymbol{u})$ is a skew tensor and $\boldsymbol{u}$ is its axial vector:

$$(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{u}=\mathbf{W}(\boldsymbol{u})\boldsymbol{v}\cdot\boldsymbol{u}=\boldsymbol{v}\cdot\mathbf{W}(\boldsymbol{u})^\top\boldsymbol{u}=-\boldsymbol{v}\cdot\mathbf{W}(\boldsymbol{u})\boldsymbol{u}=-\boldsymbol{v}\cdot\mathbf{0}=0;$$
$$\mathbf{W}(\boldsymbol{u})\boldsymbol{v}\cdot\boldsymbol{v}=\boldsymbol{v}\cdot\mathbf{W}(\boldsymbol{u})^\top\boldsymbol{v}=-\boldsymbol{v}\cdot\mathbf{W}(\boldsymbol{u})\boldsymbol{v},\ \text{whence } (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{v}=\mathbf{W}(\boldsymbol{u})\boldsymbol{v}\cdot\boldsymbol{v}=0.$$

For the second property, we call $\boldsymbol{n}$ the unit vector $\frac{\boldsymbol{u}}{|\boldsymbol{u}|}$, we consider any two mutually orthogonal unit vector $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in the plane $\operatorname{span}(\boldsymbol{b}_1,\boldsymbol{b}_2)$ orthogonal to $\boldsymbol{n}$ and we suppose them to be oriented such that $\boldsymbol{b}_1\otimes\boldsymbol{b}_2=\boldsymbol{n}$. Then

$$\mathbf{W}(\boldsymbol{u})=|\boldsymbol{u}|(\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1).$$

Indeed,

$$|\boldsymbol{u}|(\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1)\boldsymbol{u}=|\boldsymbol{u}|((\boldsymbol{b}_2\cdot\boldsymbol{u})\boldsymbol{b}_1-(\boldsymbol{b}_1\cdot\boldsymbol{u})\boldsymbol{b}_2)=0$$

because both $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are orthogonal to $\boldsymbol{u}$, and

$$\begin{aligned}|\boldsymbol{u}|^2|\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1|^2&=|\boldsymbol{u}|^2\operatorname{tr}((\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1)^\top(\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1))=\\&=|\boldsymbol{u}|^2\operatorname{tr}((\boldsymbol{b}_1\otimes\boldsymbol{b}_2)^\top(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)-(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)^\top(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)-\\&\qquad-(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)^\top(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)+(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)^\top(\boldsymbol{b}_2\otimes\boldsymbol{b}_1))=\\&=|\boldsymbol{u}|^2\operatorname{tr}((\boldsymbol{b}_2\otimes\boldsymbol{b}_1)(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)-(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)-\\&\qquad-(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)+(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)(\boldsymbol{b}_2\otimes\boldsymbol{b}_1))=\\&=|\boldsymbol{u}|^2\operatorname{tr}(\boldsymbol{b}_1\otimes\boldsymbol{b}_1+\boldsymbol{b}_2\otimes\boldsymbol{b}_2)=2|\boldsymbol{u}|^2.\end{aligned}$$

Let $\mathbf{P}_{||}(\boldsymbol{n}):=\boldsymbol{n}\otimes\boldsymbol{n}$ be the projection along the direction of $\boldsymbol{n}$, and let $\mathbf{P}(\boldsymbol{n}):=I-\mathbf{P}_{||}(\boldsymbol{n})=\boldsymbol{b}_1\otimes\boldsymbol{b}_1+\boldsymbol{b}_2\otimes\boldsymbol{b}_2$ be the projection onto the plane $\operatorname{span}(\boldsymbol{b}_1,\boldsymbol{b}_2)$.

$$\begin{aligned}\mathbf{W}(\boldsymbol{u})^2=\mathbf{W}(\boldsymbol{u})\mathbf{W}(\boldsymbol{u})&=|\boldsymbol{u}|^2(\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1)(\boldsymbol{b}_1\otimes\boldsymbol{b}_2-\boldsymbol{b}_2\otimes\boldsymbol{b}_1)\\&=|\boldsymbol{u}|^2(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)-(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)-\\&\qquad-(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)(\boldsymbol{b}_1\otimes\boldsymbol{b}_2)+(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)(\boldsymbol{b}_2\otimes\boldsymbol{b}_1)=\\&=\frac{|\mathbf{W}(\boldsymbol{u})|^2}{2}(-\boldsymbol{b}_1\otimes\boldsymbol{b}_1-\boldsymbol{b}_2\otimes\boldsymbol{b}_2)=-\frac{|\mathbf{W}(\boldsymbol{u})|^2}{2}\mathbf{P}(\boldsymbol{n}).\end{aligned}$$

Then

$$\begin{aligned}|\boldsymbol{u}\times\boldsymbol{v}|=|\mathbf{W}(\boldsymbol{u})\boldsymbol{v}|&=\sqrt{\mathbf{W}(\boldsymbol{u})\boldsymbol{v}\cdot\mathbf{W}(\boldsymbol{u})\boldsymbol{v}}=\sqrt{\boldsymbol{v}\cdot\mathbf{W}(\boldsymbol{u})^\top\mathbf{W}(\boldsymbol{u})\boldsymbol{v}}=\\&=\sqrt{-\boldsymbol{v}\cdot\mathbf{W}^2(\boldsymbol{u})\boldsymbol{v}}=\sqrt{\boldsymbol{v}\cdot\frac{|\mathbf{W}(\boldsymbol{u})|^2}{2}\mathbf{P}(\boldsymbol{n})\boldsymbol{v}}=\frac{|\mathbf{W}(\boldsymbol{u})|}{\sqrt{2}}\sqrt{\boldsymbol{v}\cdot\mathbf{P}(\boldsymbol{n})\boldsymbol{v}}.\end{aligned}$$

Since $\mathbf{P}(\boldsymbol{n})\boldsymbol{v}$ is the projection of $\boldsymbol{v}$ onto the plane orthogonal to $\boldsymbol{u}$, we have

$$\boldsymbol{v}\cdot\mathbf{P}(\boldsymbol{n})\boldsymbol{v}=(\mathbf{P}_{||}(\boldsymbol{n})\boldsymbol{v}+\mathbf{P}(\boldsymbol{n})\boldsymbol{v})\cdot\mathbf{P}(\boldsymbol{n})\boldsymbol{v}=\mathbf{P}(\boldsymbol{n})\boldsymbol{v}\cdot\mathbf{P}(\boldsymbol{n})\boldsymbol{v}=|\mathbf{P}(\boldsymbol{n})\boldsymbol{v}|^2,$$

and $|\mathbf{P}(\boldsymbol{n})\boldsymbol{v}|=|\boldsymbol{v}|\sin\vartheta_{\boldsymbol{n},\boldsymbol{v}}=|\boldsymbol{v}|\sin\vartheta_{\boldsymbol{u},\boldsymbol{v}}$. Hence

$$|\boldsymbol{u}\times\boldsymbol{v}|=\frac{|\mathbf{W}(\boldsymbol{u})|}{\sqrt{2}}\sqrt{|\mathbf{P}(\boldsymbol{n})\boldsymbol{v}|^2}=|\boldsymbol{u}||\boldsymbol{v}|\sin\vartheta_{\boldsymbol{u},\boldsymbol{v}}.$$

4. Determinant

Definition 4. A *skew trilinear form* is any map $\alpha\colon \mathcal{V}\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ such that

(1) α is linear in each argument
(2) $\alpha(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -\alpha(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) = -\alpha(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u}) = -\alpha(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}) \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathcal{V}$.

An example of skew trilinear form is the *triple product* (also called *mixed product*):

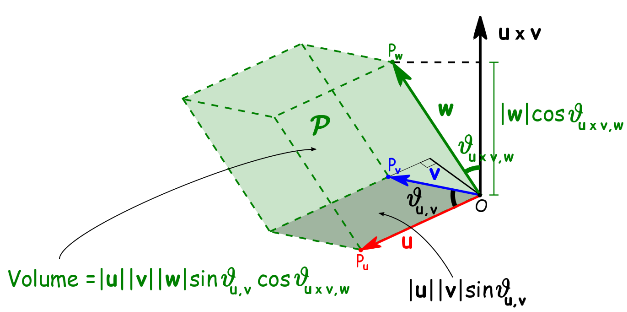
$$\beta\colon (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \mapsto \boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w}.$$

The linearity property follows immediately from the linearity of both the inner and the exterior products. Moreover we have

$$\begin{aligned} -\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w} &= (-\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w} = \boldsymbol{v}\times\boldsymbol{u}\cdot\boldsymbol{w} \\ &= -\sum_{i,j,k=1}^{3}\varepsilon_{ijk}u_j v_k w_i = \sum_{i,j,k=1}^{3}\varepsilon_{kji}u_j v_k w_i = \boldsymbol{u}\times\boldsymbol{w}\cdot\boldsymbol{v} \\ &= -\sum_{i,j,k=1}^{3}\varepsilon_{ijk}u_j v_k w_i = \sum_{i,j,k=1}^{3}\varepsilon_{jik}u_j v_k w_i = \boldsymbol{w}\times\boldsymbol{v}\cdot\boldsymbol{u}. \end{aligned}$$

We call $\mathcal{P}$ the prism in $\mathbb{R}^3$ "described by" $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$, that is built in this way:

- take the origin O of $\mathbb{R}^3$ and the points $P_{\boldsymbol{u}} := O+\boldsymbol{u}$, $P_{\boldsymbol{v}} := O+\boldsymbol{v}$ and $P_{\boldsymbol{w}} := O+\boldsymbol{w}$;
- three edges of $\mathcal{P}$ are the segments $OP_{\boldsymbol{u}}$, $OP_{\boldsymbol{v}}$ and $OP_{\boldsymbol{w}}$;
- the others edges are already completely determined (because $\mathcal{P}$ is a prism).



As we can see in the picture above, the absolute value $|\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w}|$ of $\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w}$ is the volume of the prism $\mathcal{P}$. Indeed:

$$\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w} = |\boldsymbol{u}\times\boldsymbol{v}||\boldsymbol{w}|\sin\vartheta_{\boldsymbol{u}\times\boldsymbol{v},\boldsymbol{w}} = |\boldsymbol{u}||\boldsymbol{v}|\cos\vartheta_{\boldsymbol{u},\boldsymbol{v}}|\boldsymbol{w}|\sin\vartheta_{\boldsymbol{u}\times\boldsymbol{v},\boldsymbol{w}}$$

Definition 5. The *determinant* of a tensor $\mathbf{L}$ is the real number $\det\mathbf{L}$ such that

$$\alpha(\mathbf{L}\boldsymbol{u},\mathbf{L}\boldsymbol{v},\mathbf{L}\boldsymbol{w}) = \det\mathbf{L}\,\alpha(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$$

for all vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ and for every skew trilinear form α which is not the null form. It turns out that $\det\mathbf{L} = \det[L]$, for any basis $\mathcal{B}$.

In particular, we can fix any three *linearly independent* vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ and we can consider their triple product $\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w}$. Then, for each tensor $\mathbf{L}$:

$$\det\mathbf{L} = \frac{\mathbf{L}\boldsymbol{u}\times\mathbf{L}\boldsymbol{v}\cdot\mathbf{L}\boldsymbol{w}}{\boldsymbol{u}\times\boldsymbol{v}\cdot\boldsymbol{w}}.$$

In the same way as we did for the prism $\mathcal{P}$, we call $\mathbf{L}\mathcal{P}$ the prism in $\mathbb{R}^3$ "described by" $\mathbf{L}\boldsymbol{u}, \mathbf{L}\boldsymbol{v}, \mathbf{L}\boldsymbol{w}$. Geometrically, we have that

(1) (whenever $\mathbf{L}$ is invertible) the sign of $\det \mathbf{L}$ is positive if the orientation of $(\mathbf{L}\boldsymbol{u}, \mathbf{L}\boldsymbol{v}, \mathbf{L}\boldsymbol{w})$ is the same as the orientation of $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ and negative otherwise;

(2) the absolute value of $\det \mathbf{L}$ is the ratio between the volume of the prism $\mathbf{L}\mathcal{P}$ and the prism $\mathcal{P}$:

$$|\det \mathbf{L}| = \frac{|\mathbf{L}\boldsymbol{u} \times \mathbf{L}\boldsymbol{v} \cdot \mathbf{L}\boldsymbol{w}|}{|\boldsymbol{u} \times \boldsymbol{v} \cdot \boldsymbol{w}|} = \frac{\mathrm{vol}(\mathbf{L}\mathcal{P})}{\mathrm{vol}(\mathcal{P})}.$$

The last property tells us that $|\det \mathbf{L}|$ can be geometrically interpreted as a *volume dilation factor*.
Sometimes it can be useful to take as $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ the elements $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of the basis $\mathcal{B}$. In this case, the prism $\mathcal{P}$ is a unit cube and its volume $|\boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3| = \boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3$ is 1. Then

$$\det \mathbf{L} = \mathbf{L}\boldsymbol{e}_1 \times \mathbf{L}\boldsymbol{e}_2 \cdot \mathbf{L}\boldsymbol{e}_3.$$

(If $\mathcal{B}$ is not positively oriented, then $\boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = -1$ and $\det \mathbf{L} = -\mathbf{L}\boldsymbol{e}_1 \times \mathbf{L}\boldsymbol{e}_2 \cdot \mathbf{L}\boldsymbol{e}_3$.)

Moreover, since $\mathbf{L}^{-1}$ maps $\mathbf{L}\boldsymbol{e}_1, \mathbf{L}\boldsymbol{e}_2, \mathbf{L}\boldsymbol{e}_3$ back to, respectively, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, we can immediately check that

$$\det \mathbf{L}^{-1} = \frac{\mathbf{L}^{-1}\mathbf{L}\boldsymbol{e}_1 \times \mathbf{L}^{-1}\mathbf{L}\boldsymbol{e}_2 \cdot \mathbf{L}^{-1}\mathbf{L}\boldsymbol{e}_3}{\mathbf{L}\boldsymbol{e}_1 \times \mathbf{L}\boldsymbol{e}_2 \cdot \mathbf{L}\boldsymbol{e}_3} = \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3}{\mathbf{L}\boldsymbol{e}_1 \times \mathbf{L}\boldsymbol{e}_2 \cdot \mathbf{L}\boldsymbol{e}_3} = \frac{1}{\det \mathbf{L}}.$$

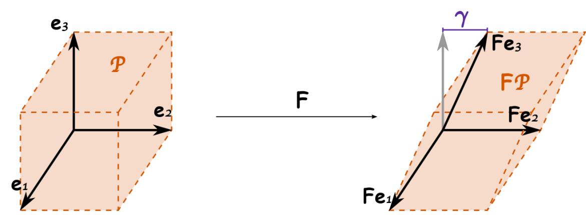
Example 4. The projection $\mathbf{P}(\boldsymbol{e}_1)$ is not invertible and "squeezes" the unit cube described by $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ into the unit square described by $\boldsymbol{e}_2, \boldsymbol{e}_3$: its determinant, indeed, is

$$\begin{aligned}\det \mathbf{L} &= \mathbf{P}(\boldsymbol{e}_1)\boldsymbol{e}_1 \times \mathbf{P}(\boldsymbol{e}_1)\boldsymbol{e}_2 \cdot \mathbf{P}(\boldsymbol{e}_1)\boldsymbol{e}_3 = \\ &= (\boldsymbol{e}_2 \otimes \boldsymbol{e}_2\boldsymbol{e}_1 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\boldsymbol{e}_1) \times (\boldsymbol{e}_2 \otimes \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\boldsymbol{e}_2) \cdot \\ &\qquad \cdot (\boldsymbol{e}_3 \otimes \boldsymbol{e}_3\boldsymbol{e}_3 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2\boldsymbol{e}_3) = \\ &= \mathbf{0} \times \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = \mathbf{0} \times \boldsymbol{e}_3 = 0.\end{aligned}$$

Example 5. The tensor $\mathbf{L}$ defined in the Example 3 has

$$\begin{aligned}\det \mathbf{L} &= \mathbf{L}\boldsymbol{e}_1 \times \mathbf{L}\boldsymbol{e}_2 \cdot \mathbf{L}\boldsymbol{e}_3 = (2\boldsymbol{e}_1 + 2\boldsymbol{e}_2) \times \left(-\frac{1}{2}\boldsymbol{e}_1 + \boldsymbol{e}_2\right) \cdot \boldsymbol{e}_3 = \\ &= 3\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 3.\end{aligned}$$

Example 6. We consider the *shear tensor* $\mathbf{F} := \mathbf{I} + \gamma \boldsymbol{e}_2 \otimes \boldsymbol{e}_3$, with $0 < \gamma \in \mathbb{R}$. (A "shear" is a strain in which parallel layers are laterally shifted.)
Since $\mathbf{F}\boldsymbol{e}_1 = \boldsymbol{e}_1$, $\mathbf{F}\boldsymbol{e}_2 = \boldsymbol{e}_2$ and $\mathbf{F}\boldsymbol{e}_3 = \gamma\boldsymbol{e}_2 + \boldsymbol{e}_3$, $\mathbf{F}$ deforms the unit cube in the following way:



As we can see from the picture, $\mathbf{F}$ does not change volumes: indeed, its determinant is

$$\det \mathbf{F} = \mathbf{F}\boldsymbol{e}_1 \times \mathbf{F}\boldsymbol{e}_2 \cdot \mathbf{F}\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot (\gamma\boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{e}_3 \cdot (\gamma\boldsymbol{e}_2 + \boldsymbol{e}_3) = 1.$$

5. Adjugate

Definition 6. The *adjugate* of the invertible tensor $\mathbf{L}$ is the tensor $\mathbf{L}^*$ such that

$$\mathbf{L}^*(\boldsymbol{u} \times \boldsymbol{v}) = \mathbf{L}\boldsymbol{u} \times \mathbf{L}\boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}.$$

The adjugate can also be written as $\mathbf{L}^* = \det \mathbf{L}(\mathbf{L}^{-1})^\top = (\det \mathbf{L})\mathbf{L}^{-\top} = \det \mathbf{L}(\mathbf{L}^\top)^{-1}$.

Why? For any vector $\boldsymbol{w}$ we have

$$\boldsymbol{u} \times \boldsymbol{v} \cdot (\mathbf{L}^*)^\top \mathbf{L}\boldsymbol{w} = \mathbf{L}^*(\boldsymbol{u} \times \boldsymbol{v}) \cdot \mathbf{L}\boldsymbol{w} = \mathbf{L}\boldsymbol{u} \times \mathbf{L}\boldsymbol{v} \cdot \mathbf{L}\boldsymbol{w} = \det \mathbf{L}(\boldsymbol{u} \times \boldsymbol{v} \cdot \boldsymbol{w}),$$

whence $(\mathbf{L}^*)^\top = \det \mathbf{L}\,\mathbf{L}^{-1}$ and $\mathbf{L}^* = \det \mathbf{L}(\mathbf{L}^{-1})^\top$.

We call $\mathcal{P}a$ the parallelogram "described by" $\boldsymbol{u}$ and $\boldsymbol{v}$, and $\mathbf{L}\mathcal{P}a$ the parallelogram "described by" $\boldsymbol{Lu}$ and $\boldsymbol{Lv}$. Then $|\boldsymbol{u} \times \boldsymbol{v}|$ and $|\mathbf{L}\boldsymbol{u} \times \mathbf{L}\boldsymbol{v}|$ are, respectively, the area of $\mathcal{P}a$ and the area of $\mathbf{L}\mathcal{P}a$, and

$$\frac{\text{area}(\mathbf{L}\mathcal{P}a)}{\text{area}(\mathcal{P}a)} = \frac{|\mathbf{L}\boldsymbol{u} \times \mathbf{L}\boldsymbol{v}|}{|\boldsymbol{u} \times \boldsymbol{v}|} = \frac{|\mathbf{L}^*(\boldsymbol{u} \times \boldsymbol{v})|}{|\boldsymbol{u} \times \boldsymbol{v}|} = \left|\mathbf{L}^*\left(\frac{\boldsymbol{u} \times \boldsymbol{v}}{|\boldsymbol{u} \times \boldsymbol{v}|}\right)\right| = |\mathbf{L}^*\boldsymbol{n}|,$$

where $\boldsymbol{n} := \frac{\boldsymbol{u} \times \boldsymbol{v}}{|\boldsymbol{u} \times \boldsymbol{v}|}$ is the normal vector to the parallelogram $\mathcal{P}a$. Hence, the geometric interpretation of the adjugate of $\mathbf{L}$ is that, whenever we take a surface $\mathcal{S}$ and its normal vector $\boldsymbol{n}$, the value $|\mathbf{L}^*\boldsymbol{n}|$ is the *area dilation factor* of the surfaces parallel to $\mathcal{S}$.

Example 7. Let $\mathbf{F} := \mathbf{I} + \gamma\boldsymbol{e}_2 \otimes \boldsymbol{e}_3$ be the shear tensor as in the Example 6. The matrices which represents $\mathbf{F}$, $\mathbf{F}^{-1}$ and $\mathbf{F}^*$ w.r.t. $\mathcal{B}$ are

$$[F] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \qquad [F]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\gamma \\ 0 & 0 & 1 \end{pmatrix} \qquad [F]^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\gamma & 1 \end{pmatrix}$$

Since $\mathbf{F}^*\boldsymbol{e}_1 = \boldsymbol{e}_1$ and $\mathbf{F}^*\boldsymbol{e}_3 = \boldsymbol{e}_3$, the tensor $\mathbf{F}$ does not change the areas of the surfaces parallel to either the vertical plane $\text{span}(\boldsymbol{e}_1, \boldsymbol{e}_2)$ or the horizontal plane span($\boldsymbol{e}_2, \boldsymbol{e}_3$). For the surfaces parallel to the vertical plane span($\boldsymbol{e}_1, \boldsymbol{e}_3$), instead, $\mathbf{F}$ makes the areas increase of a factor

$$|\mathbf{F}^* \boldsymbol{e}_2| = |\boldsymbol{e}_2 - \gamma \boldsymbol{e}_3| = \sqrt{1 + \gamma^2}.$$