FINAL EXAM – ANALYSIS FOR APPLICATIONS
JANUARY 23RD, 2019

SOLUTIONS

Exercise 1 (3 points)**.** Answer the following questions:

(1) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on $\mathbb{R}$ but *not* differentiable at $x_0 = 0$.

$$f(x) = |x| \, .$$

(2) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x \to 2} f(x) = 1$, but f is *not* continuous at $x_0 = 2$.

$$f(x) = \begin{cases} 1 & x \neq 2 \\ 0 & x = 2 \end{cases} \, .$$

(3) Can you give an example of a function which is differentiable but not continuous?
No, because if f is differentiable at x_0, then it is continuous at x_0.

Exercise 2 (2 points)**.** Find the Taylor polynomial of degree 4 of the function

$$f(x) = (1 + x)^6$$

at the point $x_0 = 0$.
To avoid long computations, it suffices to observe that

$$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \, .$$

Thus the Taylor polynomial is

$$P_4(x) = 1 + 6x + 15x^2 + 20x^3 + 15x^4 \, .$$

Exercise 3 (3 points)**.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $f(x_0) \neq 0$. Let $g = 1/f$. *By applying the definition of derivative*, prove that

$$g'(x_0) = -\frac{f'(x_0)}{f(x_0)^2} \, .$$

From the definition of derivative, we have

$$\begin{aligned} g'(x_0) &= \lim_{x \to x_0} \frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x - x_0} \\ &= \lim_{x \to x_0} \frac{\frac{f(x_0) - f(x)}{f(x) f(x_0)}}{x - x_0} \\ &= -\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} \frac{1}{f(x) f(x_0)} \\ &= -f'(x_0) \cdot \frac{1}{f(x_0) f(x_0)} = -\frac{f'(x_0)}{f(x_0)^2} \, . \end{aligned}$$

Exercise 4 (3 points). Find the derivative of the functions $f(x) = \tan(x)$ and $g(x) = \arctan(x)$. Provide the details of your reasoning.

(1) $f'(x) = 1 + \tan^2(x)$ because $f(x) = \sin(x)/\cos(x)$ and therefore:

$$f'(x_0) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = 1 + \tan(x)^2 .$$

(2) $g'(x) = \frac{1}{1+x^2}$ because g is the inverse function of f and therefore, if $y_0 = f(x_0) = \tan(x_0)$, then:

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{1+\tan(x_0)^2} = \frac{1}{1+y_0^2} .$$

Exercise 5 (5 points). Let f be the function defined by

$$f(x) = \frac{x^2}{2(1-x^2)} .$$

(1) Write the domain of definition of f.
Since $1 - x^2 = (1+x)(1-x)$, the domain of definition is $\mathbb{R} \setminus \{-1, 1\}$.

(2) Find the limits of f at $+\infty$ and $-\infty$.
We have

$$\lim_{x\to+\infty} \frac{x^2}{2(1-x^2)} = \lim_{x\to+\infty} \frac{x^2}{2x^2(\frac{1}{x^2}-1)} = \lim_{x\to+\infty} \frac{1}{2\left(\frac{1}{x^2}-1\right)} = -\frac{1}{2}$$

and analogously

$$\lim_{x\to-\infty} \frac{x^2}{1-x^2} = -\frac{1}{2} .$$

(3) Find the intervals on which f is monotone increasing and monotone decreasing.
Let us compute the first derivative:

$$f'(x) = \frac{2x(1-x^2) - x^2(-2x)}{2(1-x^2)^2} = \frac{x}{(1-x^2)^2} .$$

Since $(1-x^2)^2 \geq 0$, $f'(x) \geq 0$ if and only if $x \geq 0$, and $f'(x) \leq 0$ if and only if $x \leq 0$. Therefore f is decreasing on $(-\infty, -1) \cup (-1, 0)$ and increasing on $(0, 1) \cup (1, +\infty)$.

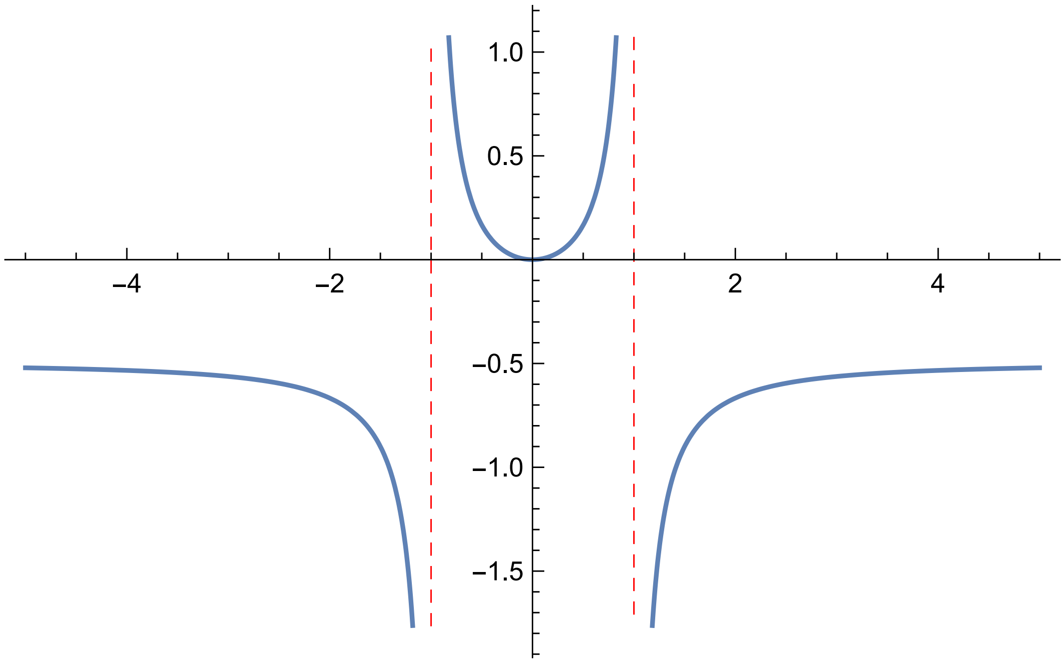
(4) Find the maxima and minima of f.
Since f' is changing sing from negative to positive at $x = 0$, $x = 0$ is the unique minimum point

(5) Find the intervals on which f is convex and concave.
Let us compute the second derivative:

$$f''(x) = \frac{(1-x^2)^2 - x \cdot 2(1-x^2)\cdot(-2x)}{(1-x^2)^4} = \frac{1 - 2x^2 + x^4 + 4x^2 - 4x^4}{(1-x^2)^4}$$
$$= \frac{-3x^4 + 2x^2 + 1}{(1-x^2)^4} = \frac{(1-x^2)(3x^2+1)}{(1-x^2)^4} = \frac{3x^2+1}{(1-x^2)^3} .$$

Since $3x^2 + 1 > 0$, we have $f''(x) > 0 \Leftrightarrow (1-x^2)^3 > 0 \Leftrightarrow 1 - x^2 > 0$, which happens exactly for $x \in (-1, 1)$. In conclusion, f is convex in the interval $(-1, 1)$, and concave in $(-\infty, -1) \cup (1, +\infty)$.

(6) Sketch a picture of the graph of f.



Exercise 6 (2 points)**.** Find the following limit. Provide the details of your reasoning.

$$\lim_{x\to 0} \frac{\ln(x+1)-x}{x^2} = -1$$

because, by Taylor's theorem,

$$\ln(x+1) = x - x^2 + O(x^3)\ .$$

Therefore

$$\ln(x+1) - x = -x^2 + O(x^3)$$

and thus:

$$\lim_{x\to 0} \frac{\ln(x+1)-x}{x^2} = \lim_{x\to 0} \frac{-x^2+O(x^3)}{x^2} = \lim_{x\to 0} (-1+O(x)) = -1\ .$$

Alternatively, use L'Hôpital's rule.

Exercise 7 (2 points)**.** The goal of this exercise is to find approximations of the solution of the equation

$$x^2 = 3\ .$$

For this purpose, let $f(x) = x^2 - 3$ and consider the equation $f(x) = 0$.

(1) Run the first 4 iterates of the algorithm of the Intermediate Value Theorem in the interval $[0, 4]$.

The algorithm runs as follows.

- Since $f(0) = -3 < 0$ and $f(4) = 13 > 0$, take the midpoint $x_1 = 2$.
- Since $f(x_1) = 2^2 - 3 = 1 > 0$, consider the new interval $[0, 2]$ and take its midpoint $x_2 = 1$.

- Since $f(x_2) = 1^2 - 3 = -2 < 0$, consider the new interval $[1, 2]$ and take its midpoint $x_3 = 3/2$.
- Since $f(x_3) = (3/2)^2 - 3 = -3/4 < 0$, consider the new interval $[3/2, 2]$ and take its midpoint $x_3 = 7/4 = 1.75$.

(2) Run the first 3 iterates of Newton's method with initial point $x_0 = 3$.

Observe that $f'(x) = 2x$. Then the algorithm runs as follows.

- The first step takes
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{6}{6} = 3 - 1 = 2\,.$$
- The second step takes
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{1}{4} = \frac{7}{4}\,.$$
- The third step takes
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{7}{4} - \frac{\frac{49}{16} - 3}{\frac{7}{2}} == \frac{7}{4} - \frac{1}{56} = \frac{97}{56} \approx 1.7321\,.$$
Observe that this is a much better approximation in only three steps, since the exact solution is
$$\sqrt{3} \approx 1.7320$$