FINAL EXAM – ANALYSIS FOR APPLICATIONS
JANUARY 23RD, 2019

SOLUTIONS

Exercise 1 (3 points)**.** Answer the following questions:

(1) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 3} f(x) = -1$, but f is *not* continuous at $x_0 = 3$.

$$f(x) = \begin{cases} -1 & x \neq 3 \\ 0 & x = 3 \end{cases} .$$

(2) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on $\mathbb{R}$ but *not* differentiable at $x_0 = 0$.

$$f(x) = |x| .$$

(3) Can you give an example of a function which is differentiable but not continuous?
No, because if f is differentiable at x_0, then it is continuous at x_0.

Exercise 2 (2 points)**.** Find the Taylor polynomial of degree 3 of the function

$$f(x) = x\ln(1+x)$$

at the point $x_0 = 0$.
To avoid long computations, it suffices to observe that the Taylor expansion of $\ln(1+x)$ is

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots$$

Thus the Taylor polynomial of $\ln(1+x)$ is

$$P_3(x) = x^2 - \frac{1}{2}x^3 .$$

Exercise 3 (3 points)**.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Let $g = f^2$. *By applying the definition of derivative*, prove that

$$g'(x_0) = 2f(x_0)f'(x_0) .$$

From the definition of derivative, we have

$$\begin{aligned} g'(x_0) &= \lim_{x\to x_0} \frac{f(x)^2 - f(x_0)^2}{x - x_0} \\ &= \lim_{x\to x_0} \frac{(f(x) - f(x_0))(f(x) + f(x_0))}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x\to x_0} (f(x) + f(x_0)) \\ &= f'(x_0) \cdot (f(x_0) + f(x_0)) = 2f(x_0)f'(x_0) . \end{aligned}$$

Exercise 4 (3 points)**.** Find the derivative of the functions $f(x) = \tanh(x)$ and $g(x) = \text{arctanh}(x)$. Provide the details of your reasoning.

(1) $f'(x) = 1 - \tanh^2(x)$ because $f(x) = \sinh(x)/\cosh(x)$ and therefore:

$$f'(x_0) = \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)} = 1 - \tanh(x)^2 \, .$$

(2) $g'(x) = \frac{1}{1-x^2}$ because g is the inverse function of f and therefore, if $y_0 = f(x_0) = \tanh(x_0)$, then:

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{1 - \tanh(x_0)^2} = \frac{1}{1 - y_0^2} \, .$$

Exercise 5 (5 points)**.** Let f be the function defined by

$$f(x) = \frac{x^2}{8(x^2 - 4)} \, .$$

(1) Write the domain of definition of f.
Since $x^2 - 4 = (x + 2)(x - 2)$, the domain of definition is $\mathbb{R} \setminus \{-2, 2\}$.

(2) Find the limits of f at $+\infty$ and $-\infty$.
We have

$$\lim_{x\to+\infty} \frac{x^2}{8(x^2 - 4)} = \lim_{x\to+\infty} \frac{x^2}{8x^2(1 - \frac{4}{x^2})} = \lim_{x\to+\infty} \frac{1}{8\,(1 - \frac{4}{x^2})} = \frac{1}{8}$$

and analogously

$$\lim_{x\to-\infty} \frac{x^2}{8(x^2 - 4)} = \frac{1}{8} \, .$$

(3) Find the intervals on which f is monotone increasing and monotone decreasing.
Let us compute the first derivative:

$$f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{8(x^2 - 4)^2} = -\frac{x}{(x^2 - 4)^2} \, .$$

Since $(x^2 - 4)^2 \geq 0$, $f'(x) \geq 0$ if and only if $x \leq 0$, and $f'(x) \leq 0$ if and only if $x \geq 0$. Therefore f is increasing on $(-\infty, -2) \cup (-2, 0)$ and decreasing on $(0, 2) \cup (2, +\infty)$.

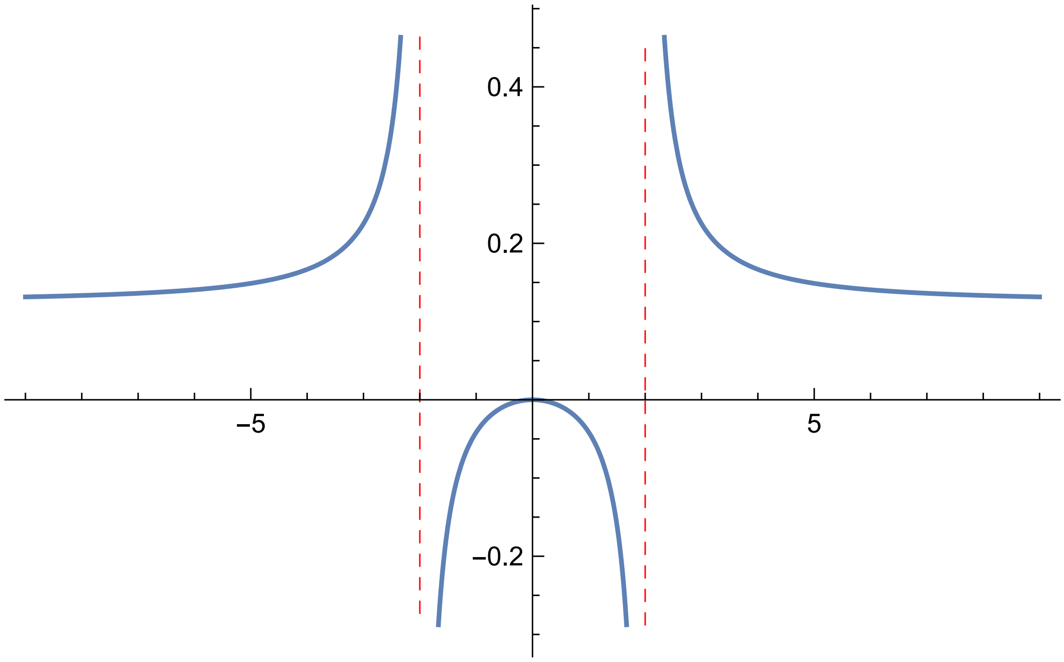
(4) Find the maxima and minima of f.
Since f' is changing sing from positive to negative at $x = 0$, $x = 0$ is the unique maximum point

(5) Find the intervals on which f is convex and concave.
Let us compute the second derivative:

$$f''(x) = -\frac{(x^2 - 4)^2 - x \cdot 2(x^2 - 4) \cdot (2x)}{(x^2 - 4)^4} = -\frac{x^4 - 8x + 16 - 4x^4 + 16x^2}{(x^2 - 4)^4}$$
$$= \frac{3x^4 - 8x^2 - 16}{(x^2 - 1)^4} = \frac{(x^2 - 4)(3x^2 + 4)}{(x^2 - 4)^4} = \frac{3x^2 + 4}{(x^2 - 4)^3} \, .$$

Since $3x^2 + 4 > 0$, we have $f''(x) > 0 \Leftrightarrow (x^2 - 4)^3 > 0 \Leftrightarrow x^2 - 4 > 0$, which happens exactly for $|x| > 2$. In conclusion, f is concave in the interval $(-2, 2)$, and convex in $(-\infty, -2) \cup (2, +\infty)$.

(6) Sketch a picture of the graph of f.



Exercise 6 (2 points)**.** Find the following limit. Provide the details of your reasoning.

$$\lim_{x\to 0} \frac{2\cos(x) - 2 + x^2}{x^4} = \frac{1}{12}$$

because, by Taylor's theorem,

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5)\,.$$

Therefore

$$2\cos(x) - 2 + x^2 = \frac{x^4}{12} + O(x^5)$$

and thus:

$$\lim_{x\to 0} \frac{2\cos(x) - 2 + x^2}{x^4} = \lim_{x\to 0} \frac{\frac{x^4}{12} + O(x^5)}{x^4} = \lim_{x\to 0} \left(\frac{1}{12} + O(x)\right) = \frac{1}{12}\,.$$

Alternatively, use L'Hôpital's rule.

Exercise 7 (2 points)**.** The goal of this exercise is to find approximations of the solution of the equation

$$x^2 = 3\,.$$

For this purpose, let $f(x) = x^2 - 3$ and consider the equation $f(x) = 0$.

(1) Run the first 4 iterates of the algorithm of the Intermediate Value Theorem in the interval $[0, 4]$.

The algorithm runs as follows.

- Since $f(0) = -3 < 0$ and $f(4) = 13 > 0$, take the midpoint $x_1 = 2$.

- Since $f(x_1) = 2^2 - 3 = 1 > 0$, consider the new interval $[0, 2]$ and take its midpoint $x_2 = 1$.
- Since $f(x_2) = 1^2 - 3 = -2 < 0$, consider the new interval $[1, 2]$ and take its midpoint $x_3 = 3/2$.
- Since $f(x_3) = (3/2)^2 - 3 = -3/4 < 0$, consider the new interval $[3/2, 2]$ and take its midpoint $x_3 = 7/4 = 1.75$.

(2) Run the first 3 iterates of Newton's method with initial point $x_0 = 3$.

Observe that $f'(x) = 2x$. Then the algorithm runs as follows.

- The first step takes
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{6}{6} = 3 - 1 = 2\,.$$
- The second step takes
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{1}{4} = \frac{7}{4}\,.$$
- The third step takes
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{7}{4} - \frac{\frac{49}{16} - 3}{\frac{7}{2}} == \frac{7}{4} - \frac{1}{56} = \frac{97}{56} \approx 1.7321\,.$$
Observe that this is a much better approximation in only three steps, since the exact solution is
$$\sqrt{3} \approx 1.7320$$