MINIMAL SURFACES IN HYPERBOLIC SPACE WITH "SMALL" QUASICIRCLE AT INFINITY

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SOME FACTS ON UNIVERSAL TEICHMÜLLER SPACE...

- A *quasiconformal mapping* $f : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is a homeomorphism, absolutely continuous on lines, such that df maps circles to ellipses of bounded eccentricity.
- The *quasiconformal distortion* $K(f)$ of $f : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is the essential supremum of the ratio of major radius to minor radius of such ellipses, taken over all points of $\mathbb{C}\cup\{\infty\}$.
- *Universal Teichmüller space* $\mathcal{T}(\mathbb{D})$ is defined as the space of quasiconformal mappings $f : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ which are conformal on $\mathbb{D} = \{|z| < 1\}$, up to the equivalence relation

$$f \sim f' \Leftrightarrow \text{ there exists } A \in \mathrm{PSL}(2,\mathbb{C}) \text{ such that } f'|_{\partial\mathbb{D}} = A \circ f|_{\partial\mathbb{D}}.$$

- A simple closed curve in $\mathbb{C}$ is a K-*quasicircle* if it is the image of $\partial\mathbb{D}$ under some $f \in \mathcal{T}(\mathbb{D})$ with $K(f) \leq K$.
- The *Teichmüller distance* on $\mathcal{T}(\mathbb{D})$ is defined as

$$d_{\mathcal{T}}([\bar{f}],[\bar{g}]) = \frac{1}{2}\inf_{f\sim\bar{f},\, g\sim\bar{g}} \log K(g^{-1}\circ f).$$

- The *Bers embedding* $\mathcal{B}$ maps $\mathcal{T}(\mathbb{D})$ to the space of holomorphic quadratic differentials on $\mathbb{D}$, by mapping $[f]$ to the Schwarzian derivative of $f|_{\mathbb{D}}$. This induces the *Bers norm* $||f|| = ||\mathcal{B}(f)||_2$ on $\mathcal{T}(\mathbb{D})$.

...AND MINIMAL SURFACES IN $\mathbb{H}^3$

- For every Jordan curve Γ in $\partial_\infty\mathbb{H}^3 \cong \mathbb{C}\cup\{\infty\}$ there exists a minimal embedded disc S in $\mathbb{H}^3$ such that its boundary at infinity coincides with Γ (Anderson 1983). Uniqueness does not hold in general.
- ⋆ If the supremum $||\lambda||_\infty$ of the principal curvatures $\pm\lambda$ of a minimal embedded disc S is in $(-1,1)$, then S is the unique minimal disc asymptotic to $\Gamma = \partial_\infty S$.
- Under the same assumption $||\lambda||_\infty \in (-1,1)$, the boundary at infinity $\Gamma = \partial_\infty S$ is a quasicircle (Epstein 1986). It tends to the identity in $\mathcal{T}(\mathbb{D})$ if $||\lambda||_\infty$ goes to zero. The main theorem of this poster is a partial converse statement.

QUASI-FUCHSIAN MANIFOLDS

Let Σ be a closed surface of genus ≥ 2.

- The limit set $\Lambda(G)$ of a quasi-Fuchsian group $G < \mathrm{PSL}(2,\mathbb{C})$, $G \cong \pi_1(\Sigma)$, is a quasicircle. Hence the lift to the universal cover of minimal surfaces in quasi-Fuchsian manifolds give examples of minimal surfaces with a quasicircle at infinity.
- A quasi-Fuchsian manifold containing a minimal surface with principal curvatures in $(-1,1)$ is called *almost-Fuchsian*. In an almost-Fuchsian manifold, the minimal surface is unique (Uhlenbeck 1983).
- G acts freely and properly discontinuously on the two connected components of $\partial_\infty\mathbb{H}^3 \setminus \Lambda(G)$. Their quotients give complex structures on S, the *conformal ends*.

THEOREM (S.)

There exist universal constants K and C such that every minimal embedded disc in $\mathbb{H}^3$ with boundary at infinity a K-quasicircle $\Gamma \subset \partial_\infty\mathbb{H}^3$ has principal curvatures bounded by $||\lambda||_\infty \leq C\log K$.

TWO COROLLARIES

The first corollary is obtained by applying Fact ⋆ to the theorem above.

COROLLARY 1: **There exists a universal constant K' such that every K'-quasicircle $\Gamma \subset \partial_\infty(\mathbb{H}^3)$ is the boundary at infinity of a unique minimal embedded disc.**

The second corollary concerns quasi-Fuchsian manifolds and is proved by considering the lift to the universal cover.

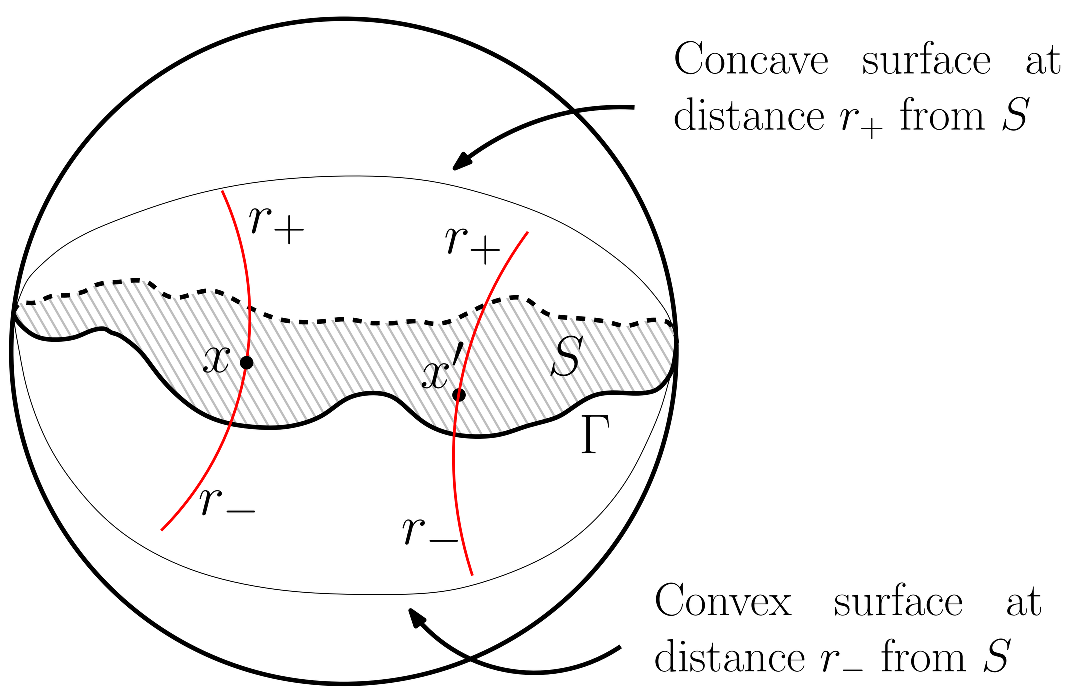
COROLLARY 2: **If the Teichmüller distance between the conformal ends of a quasi-Fuchsian manifold M is smaller than a universal constant D, then M is almost-Fuchsian.**

TRAP THE MINIMAL SURFACE IN THE CONVEX HULL

We now give some ideas of the proof of the theorem. A key property is that a minimal disc S with $\partial_\infty S = \Gamma$ is contained in the convex hull of Γ.

Given a quasicircle at infinity $\Gamma = f(\partial\mathbb{D})$ for $f \in \mathcal{T}(\mathbb{D})$, if $||f||_{\mathcal{B}} < 1/2$, there is a foliation of $\mathbb{H}^3$ by surfaces equidistant from S.

The leaves of the foliation at signed distance r from S are concave for $r \geq r_+ > 0$ (r_+ is big, in general) and convex for $r \leq r_- < 0$.



The difference $r_+ - r_-$ is estimated in terms of the Bers norm of the quasicircle at infinity:

$$r_+ - r_- \leq \mathrm{arctanh}(2||f||_{\mathcal{B}}).$$

As a consequence, given any point x of S there is a geodesic segment through x of lenght at most $\mathrm{arctanh}(2||f||_{\mathcal{B}})$ orthogonal at the endpoints to two planes P_- and P_+, such that the convex hull $\mathcal{CH}(\Gamma)$ is contained between P_- and P_+.

SCHAUDER ESTIMATES

Let S be a minimal disc in $\mathbb{H}^3$ and P be a totally geodesic plane. The function

$$u(x) = \sinh d(x,P)$$

satisfies the equation

$$\Delta u - 2u = 0$$

where Δ is the Laplace-Beltrami operator on S. One can use Schauder theory to give a uniform estimate on the second derivatives of u in normal coordinates:

$$||u||_{C^2(B(0,\frac{R}{2}))} \leq C||u||_{C^0(B(0,R))}.$$

The estimate holds for all minimal surfaces with boundary at infinity a K-quasicircle, and C only depends on K.

We know for every point x on S, S is trapped between two planes at small distance from x. Hence the second derivatives of u at every point can be estimated by the Bers norm of the quasicircle. The estimate does not depend on the point x.

Using the formula for the shape operator B of S:

$$\mathrm{Hess}u - u\mathrm{I} = \sqrt{1+u^2-||\,\mathrm{grad}\, u||^2}B$$

and applying the previous estimates (which do not depend on x and on S), one obtains the inequality

$$||\lambda||_\infty \leq \frac{C||f||_{\mathcal{B}}}{\sqrt{1-C||f||^2_{\mathcal{B}}}}$$

for minimal surfaces S with $\partial_\infty S$ a K-quasicircle, where C only depends on K. The function in the right hand side is sublinear in $||f||_{\mathcal{B}}$ in a neighborhood of 0.

ESSENTIAL BIBLIOGRAPHY

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