INFINITESIMAL DEFORMATIONS OF HYPERBOLIC METRICS ON A SURFACE AND FLAT LORENTZIAN STRUCTURES

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OUR RESULT FOR CLOSED SURFACES

MGH FLAT SPACETIMES

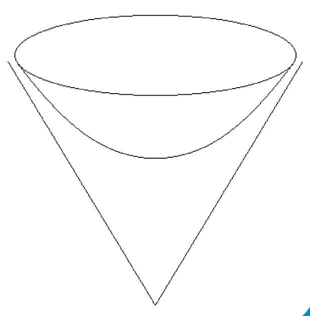
A flat Lorentzian 3-manifold M is a $(\mathbb{R}^{2,1}, \mathrm{Isom}(\mathbb{R}^{2,1}))$-structure, where $\mathbb{R}^{2,1}$ is Minkowski space.

M is **globally hyperbolic** (GH) if there exists a embedding $\sigma : S \to M$ of a Cauchy surface: every inextensible causal curve (whose tangent vector has non-positive norm at every point) in M intersects $\sigma(S)$ exactly once. In this case, M is topologically $S \times \mathbb{R}$.

If S is a closed surface of genus $g \geq 2$, the holonomy $\rho = (f, t) : \pi_1(S) \to \mathrm{SO}(2,1) \ltimes \mathbb{R}^{2,1}$ parametrizes maximal GH spacetimes homeomorphic to $S \times \mathbb{R}$ (Mess [3]).

- The linear part $f : \pi_1(S) \to \mathrm{SO}(2,1)$ is a Fuchsian holonomy;
- The translation part $t : \pi_1(S) \to \mathbb{R}^{2,1}$ is a cohomology class in $H^1_f(\pi_1(S), \mathbb{R}^{2,1})$.

EXAMPLE



$M = I^+(0)/f(\pi_1(S))$ is a MGH spacetime with $t = 0$.
When $t \neq 0$, $\tilde{M}$ is a more complicated convex domain in $\mathbb{R}^{2,1}$.

A NATURAL ISOMORPHISM

There is a $\mathrm{SO}(2,1)$-equivariant isomorphism

$$\Lambda : \mathbb{R}^{2,1} \to \mathfrak{so}(2,1) \qquad \Lambda(t) : v \mapsto t \boxtimes v\,.$$

Recall that $\mathrm{Teich}(S)$ is diffeomorphic to an open subset of $\mathrm{Hom}(\pi_1(S), \mathrm{SO}(2,1))/\mathrm{SO}(2,1)$. Its tangent space is $H^1_{\mathrm{Ad}f}(\pi_1(S), \mathfrak{so}(2,1))$ (Goldman [2]).

It follows that **flat MGH structures on $S \times \mathbb{R}$ are parametrized by the tangent bundle $T\mathrm{Teich}(S)$.**

THE DIFFERENTIAL APPROACH

By Mess [3], every MGH contains a **strictly convex** spacelike Cauchy surface. Suppose you know the embedding data of such a surface $\sigma : S \to M$, that is:

- First fundamental form: the Riemannian metric I induced on S by σ;
- Shape operator: the non-degenerate symmetric tensor $s = \nabla N : TS \to TS$, for N future unit normal.

These tensors satisfy the Gauss-Codazzi equations

$$\begin{cases} \det s = -K_I > 0 \\ d_I^{\nabla} s(v,w) = (\nabla_v s)(w) - (\nabla_w s)(v) - s[v,w] = 0 \end{cases}$$

They determine M up to isometry.

HOW TO FIND THE HOLONOMY OF M?

- The third fundamental form $h(v,w) = I(s(v), s(w))$ is a hyperbolic metric on S.
  The holonomy of h is the linear part $f : \pi_1(S) \to \mathrm{SO}(2,1)$!
- Lift $b = s^{-1}$ to the universal cover $\tilde{S} = \mathbb{H}^2 \to S$.
  There is a function u such that $\tilde{b} = u(\mathrm{id}) - \mathrm{Hess}(u)$.
  By the equivariance of $\tilde{b}$, you can find t_γ such that $u(\gamma x) - u(x) = -\langle t_\gamma, \gamma x\rangle$.
  $t : \pi_1(S) \to \mathbb{R}^{2,1}$ is the translation part!

CONCLUSION

We described MGH flat Lorentzian manifolds by pairs:

- Hyperbolic metric h on S;
- Symmetric Codazzi tensor b (satisfying $d_h^{\nabla} b = 0$ for h).

THE INFINITESIMAL DEFORMATION

A symmetric tensor b represents a tangent vector to the space of metrics on S, via

$$h(b(v), w) = \left.\frac{d}{dt}\right|_{t=0} h_t(v,w)$$

b is NOT a deformation of hyperbolic metrics:
h_t is not a hyperbolic metric for small t.
BUT, by the uniformization theorem, we project b to a genuine deformation of hyperbolic metrics.

HOW TO FIND THE HOLONOMY DEFORMATION?

Take the path of holonomies ρ_t and developing maps dev_t for the hyperbolic metrics h_t^* conformal to h_t.
Differentiate $\mathrm{dev}_t(\gamma(x)) = \rho_t(\gamma)\mathrm{dev}_t(x)$.
The obstruction to the equivariance of the vector field

$$X(x) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{dev}_t(x)$$

describes a cohomology class $\tau \in H^1_{\mathrm{Ad}f}(\pi_1(S), \mathfrak{so}(2,1))$.
This element is the deformation

$$\tau(\gamma) = \left(\left.\frac{d}{dt}\right|_{t=0} \rho_t(\gamma)\right)\rho_0(\gamma)^{-1}\,.$$

QUESTION

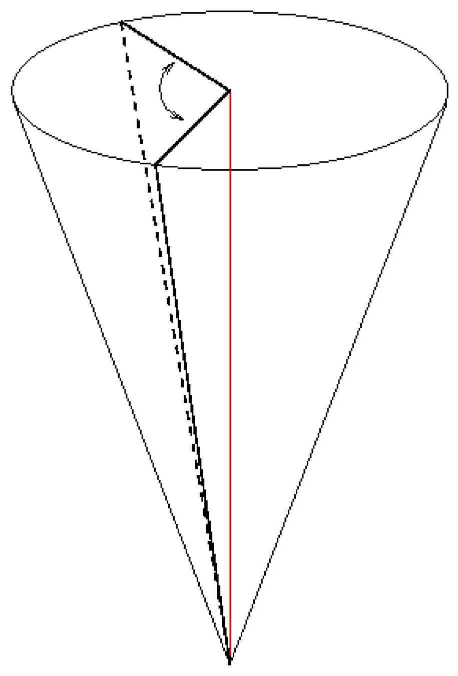
Do the translation deformation t and the infinitesimal deformation τ coincide under the isomorphism Λ?

ANSWER

Not exactly.
They differ by the complex structure of $\mathrm{Teich}(S)$.

WHAT ABOUT CONE SINGULARITIES?

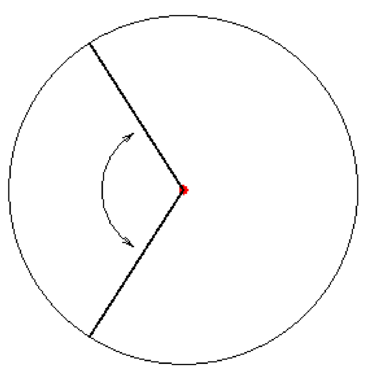
ANALOGIES AND DIFFERENCES



The local model of a particle singularity in a flat Lorentzian manifold is a wedge of angle $\theta < 2\pi$ in $\mathbb{R}^{2,1}$ around a timelike line l (particle), where the edges are glued through a rotation around l.
The holonomy representation of a path γ winding around the singular line has:

- Linear part $f(\gamma) = R_\theta$, a rotation of angle θ fixing the origin
- Translation part $t_\gamma = R_\theta(v) - v$ orthogonal to the singular line

The developing map for singular structures is not injective!
In this case, Mess' techniques do not apply directly.

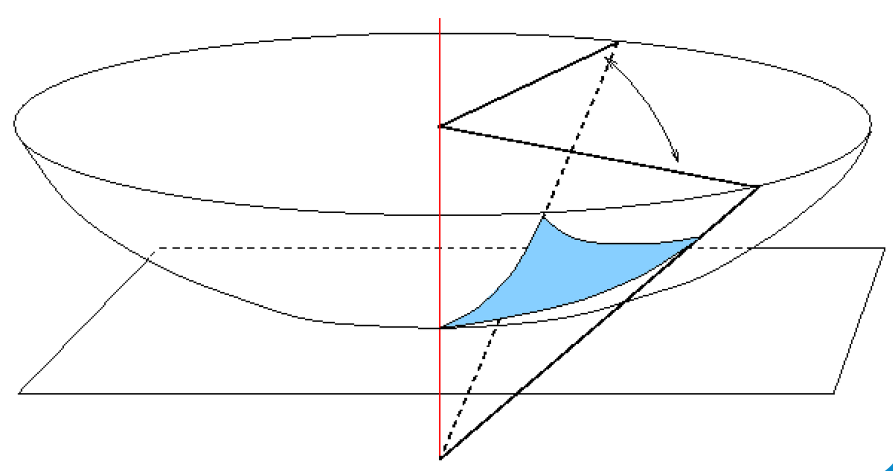


A Cauchy surface $\sigma(S)$ in a MGH flat spacetime with particles intersect the particle lines in singular points.
Given the first fundamental form I and the shape operator s, the third fundamental form $h(v,w) = I(s(v), s(w))$ is a hyperbolic metric with cone points of the same angle θ.

Again, its holonomy is the linear part of the holonomy of M!

In our differential approach, MGH spacetimes with particles are described by

- Hyperbolic metric h on S with cone singularities of angles $\theta_1, \ldots, \theta_n < 2\pi$
- Symmetric Codazzi tensor b satisfying $d_h^{\nabla} b = 0$ and $b \to \lambda_i(\mathrm{id})$ at the singularities



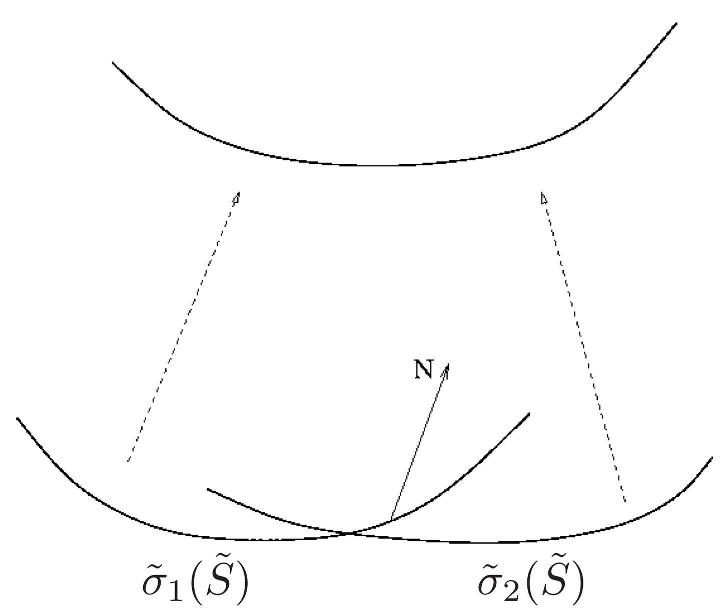
OUR THEOREM FOR SINGULAR MANIFOLDS

By using this differential approach, we can prove the following:

THEOREM

Let $f : \pi_1(S) \to \mathrm{Isom}(\mathbb{R}^{2,1})$ be the holonomy of a hyperbolic metric h with cone singularities of angles $\pi \neq \theta_1, \ldots, \theta_n < 2\pi$ at the punctures of S; let $t \in H^1(\pi_1(S), \mathbb{R}(2,1))$ a cocycle such that $t_\gamma = f(\gamma)v - v$ for some $v = v(\gamma)$ for every peripheral path $\gamma \in \pi_1(S)$. Then there exist a unique future-complete MGH flat spacetime with particle singularities $\theta_1, \ldots, \theta_n$, holonomy $\rho = (f, t)$ and third fundamental form h.

The geometric idea behind uniqueness:



Let M_1 and M_2 future complete with the same holonomy. Lift a Cauchy embedding $\sigma_i : S \to M_i$ to the universal cover $\tilde{\sigma}_i : \tilde{S} \to \tilde{M}_i$

- Any small deformation of a $\tilde{\sigma}_i$ preserving the holonomy provides an embedding into the same spacetime M_i
- It is possible to deform (in the large) both $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ to an embedding $\tilde{\sigma}_3$ far enough in the future, with the same holonomy, staying inside the same spacetime.
- Then M_1 and M_2 are the same spacetime!

COROLLARY

Future-complete MGH flat spacetimes with particles of angles $\theta_1, \ldots, \theta_n$, containing a strictly convex Cauchy surface, are parametrized by $T\mathrm{Teich}_{\theta_1,\ldots,\theta_n}(S)$.

Indeed, a simple computation shows that the first-order variation of θ_i is 0 if and only if the vector t_γ is orthogonal to the axis of rotation of $f(\gamma)$, for every peripheral path γ.

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[4] Bonsante F. and Seppi A. "Infinitesimal deformations of hyperbolic metrics on a surface and flat Lorentzian structures" *In preparation.*