EXERCISE SHEET 2

Exercise 1. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *monotone increasing* if, for every $x_1, x_2 \in \mathbb{R}$, if $x_1 < x_2$, then $f(x_1) \leq f(x_2)$. The goal of this exercise is to show that the exponential function is monotone increasing. For this purpose, proceed by the following steps:

(1) Show that, if $x > 0$, then $e^x > 1$.
(2) Show that, if $x < 0$, then $e^x < 1$. *Hint: you can use that* $e^{-x} = 1/e^x$.
(3) Show that, if $x_1 < 0 < x_2$, then $e^{x_1} < e^{x_2}$.
(4) Show that, if $0 < x_1 < x_2$, then $e^{x_1} < e^{x_2}$. *Hint: write* $x_2 = x_1 + (x_2 - x_1)$*, use that* $e^{x+y} = e^x e^y$ *and use the first point.*
(5) Show that, if $x_1 < x_2 < 0$, then $e^{x_1} < e^{x_2}$.
(6) Deduce from the previous points that the exponential is monotone increasing. *Hint: divide the proof in different cases, according to the sign of* x_1 *and* x_2.

Exercise 2. Recall that the *hyperbolic trigonometric functions* are defined by:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} .$$

(1) Show that $\cosh(-x) = \cosh(x)$ and $\sinh(-x) = -\sinh(x)$.
(2) Show that $\cosh^2(x) - \sinh^2(x) = 1$.
(3) Show directly the identities:

$$\sinh(x + y) = \cosh(x)\sinh(y) + \sinh(x)\cosh(y)$$

and

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) .$$

(4) Show that

$$\lim_{x\to+\infty} \cosh(x) = +\infty \qquad \text{and} \qquad \lim_{x\to+\infty} \sinh(x) = +\infty .$$

Hint: for the first limit, show and use that $\cosh(x) > e^x/2$*; for the second limit, use the identity* $\cosh^2(x) - \sinh^2(x) = 1$.

Exercise 3. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *even* if $f(-x) = f(x)$ for every x, while f is *odd* if $f(-x) = -f(x)$ for every x. Determine whether the following functions are even or odd:

(1) $f(x) = x^3$;
(2) $f(x) = x^4$;
(3) $f(x) = x^n$ for n even;
(4) $f(x) = x^n$ for n odd;
(5) $f(x) = e^x$;
(6) $f(x) = 1$;
(7) $f(x) = 0$;
(8) $f(x) = \sin(x)$;
(9) $f(x) = \cos(x)$;
(10) $f(x) = \sinh(x)$;
(11) $f(x) = \cosh(x)$;

Exercise 4. Prove the following properties:

(1) If $f : \mathbb{R} \to \mathbb{R}$ is an odd function, then $f(0) = 0$.
(2) If $f : \mathbb{R} \to \mathbb{R}$ is both odd and even, then $f(x) = 0$ for every x.
(3) If $f : \mathbb{R} \to \mathbb{R}$ is even, then

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) \ .$$

(4) If $f : \mathbb{R} \to \mathbb{R}$ is odd, then

$$\lim_{x \to +\infty} f(x) = - \lim_{x \to -\infty} f(x) \ .$$

Exercise 5. Show that

$$\lim_{x \to -\infty} x^n = \begin{cases} +\infty & \text{if } n \geq 1 \text{ and } n \text{ is even,} \\ -\infty & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \\ 1 & \text{if } n = 0, \\ 0 & \text{if } n \leq 0. \end{cases}$$

Hint: use the previous exercises and the limits of x^n at $+\infty$ seen in class.