EXERCISE SHEET 7

WRITTEN SOLUTIONS OF EXERCISES 1.4 AND 2.5 TO BE PRESENTED ON 20/11

Exercise 1. Apply the rule for the derivative of the inverse function in order to show that:

(1) $(\operatorname{arcsinh}(y))' = \dfrac{1}{\sqrt{1+y^2}}$.

(2) $(\operatorname{arccosh}(y))' = \dfrac{1}{\sqrt{y^2-1}}$.

(3) $(\operatorname{arctanh}(y))' = \dfrac{1}{1-y^2}$.

(4) $(\sqrt[3]{y})' = \dfrac{1}{3\sqrt[3]{y^2}}$.

Exercise 2. The objective of this exercise is to compute again the derivative of point (2) of the previous exercise, but by a different method.

(1) Show that the equation $x + 1/x = 2y$, where x is the variable and y is fixed, has:
- No solution if $y < 1$;
- One solution if $y = 1$, namely $x = 1$;
- Two solutions if $y > 1$, given by $x = y \pm \sqrt{y^2 - 1}$.

(2) Deduce that the equation $\cosh(t) = y$ has
- One solution if $y = 1$, namely $t = 0$;
- Two solutions if $y > 1$, given by $t = \log(y \pm \sqrt{y^2 - 1})$.

(*Hint: put $x = e^t$ and use the previous point.*)

(3) Deduce that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \cosh(t)$ is neither injective nor surjective.

(4) Deduce that the function $f : [0, +\infty) \to [1, +\infty)$ given by $f(t) = \cosh(t)$ is bijective, with inverse

$$f^{-1}(y) = \log(y + \sqrt{y^2 - 1}) .$$

(5) Show that

$$(f^{-1})'(y) = \frac{1}{\sqrt{y^2 - 1}}$$

by differentiating the expression of the previous point.

Exercise 3. Compute the derivatives of the following functions, on their domain of definition:

(1) $f(x) = 4\log(x) + 5x^3$
(2) $f(x) = \log(\sin(x))$
(3) $f(x) = \sqrt{x^2}$
(4) $f(x) = (\sqrt{x})^2$
(5) $f(x) = \sqrt{3x^2 + 2x - 7}$
(6) $f(x) = \arctan(1 + e^{2x})$
(7) $f(x) = 4e^x \arcsin(x)$
(8) $f(x) = \arccos(x^3 + x)$
(9) $f(x) = \log(1 + 3e^{2x})$
(10) $f(x) = \log(\log(\log(x)))$

Exercise 4. Using the definition of the power function, namely $x^\alpha := e^{\alpha \log(x)}$, show the following properties, for every $\alpha \in \mathbb{R}$:

(1) $\log(x^\alpha) = \alpha \log(x)$
(2) $x^{\alpha+\beta} = x^\alpha x^\beta$;
(3) $(x^\alpha)^\beta = x^{\alpha\beta}$.